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PRESSING A RIGID PLASTIC SURFACE BY A WEDGE-SHAPED STAMP UNDER THE COULOMB–MOHR YIELD CONDITION

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The problem of treating a surface with a wedge-shaped stamp is considered using the model of an ideal rigid plastic body. The strain fields in the vicinity of singularities of the displacement velocity field (on the discontinuity lines of the displacement velocities and at the center of the fan of characteristics) are investigated taking into account irreversible compressibility.

Key words: *rigid plastic model, plane stressed state, finite strain tensor, compressibility.*

Technological problems of plasticity theory were studied in [1, 2], but the strain field has not been investigated or has been considered using the Tresca or Mises yield conditions, which within the model of a rigid plastic body, lead to plastic incompressibility of the material. From experimental studies, it is known that material fracture is preceded by loosening. In the surface pressing process considered, material fracture is represented as surface peeling, which can be preceded by loosening of the material. These effects can be explained using the Coulomb–Mohr yield condition, which has previously been used mostly for soils.

It has been shown [3–5] that in the plastic region, the strains are distributed extremely nonuniformly, can reach large values, and are observed mainly in the vicinity of the singularities of the displacement velocity field.

In the present work, estimates are obtained for the field of finite strains in the problem of pressing a rigid plastic surface by a wedge-shaped stamp (Fig. 1).

As a measure of strains we use the Almansi finite strain tensor

$$E_{ij} = (\delta_{ij} - x^0_{k,i} x^0_{k,j})/2, \tag{1}$$

where δ_{ij} is the Kronecker symbol, x_i^0 are the Lagrangian coordinates of a particle, x_i are the Eulerian coordinates of a particle, $x^0_{j,i} = a_{ij}$ are the components of the distorsion tensor.

The yield condition for the material being compressed is taken to be the Coulomb–Mohr condition, which for plane strain is written as follows [6]:

$$(\sigma_{11} - \sigma_{22})^2/4 + \sigma_{12}^2 = (k + \cos 2\varphi\,(\sigma_{11} + \sigma_{22})/2)^2,$$

where k and φ are constants that describe the medium studied.

Let a wedge-shaped stamp move with velocity v_0 along the x_1 axis. The stress and velocity fields are analyzed in the same manner as in the problem of penetration of a wedge into a rigid plastic half-space [7].

The plastic region of the problem considered (see Fig. 1) consists of two triangular regions A_0PA_1 and A_2PA_3 with uniform stress states which are connected by the centered fan A_1PA_2 consisting of straight-line characteristics η and the family of logarithmic spirals of characteristics ξ, whose parametric equation is written as

$$x_1 = a - D\,\mathrm{e}^{-\cot\,(2\varphi)(\psi-\alpha)}\cos(\psi+\varphi) + v_0 t, \qquad x_2 = b - D\,\mathrm{e}^{-\cot\,(2\varphi)(\psi-\alpha)}\sin(\psi+\varphi).$$

Here a and b are the coordinates of the point P and ψ is the angle of inclination of the largest principal stress component to the axis x_1. For the characteristic A_1A_2, we use the constant $D = c/(2\cos\varphi)$, where c is the length of the contact region A_0P.

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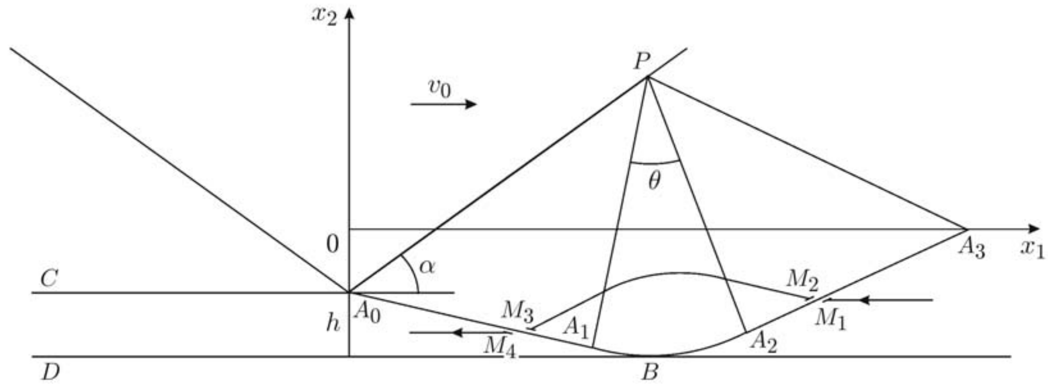

Fig. 1. Diagram of surface pressing by a wedge-shaped stamp.

The region CA_0A_1BD is the region of residual strains of depth h for motion of the wedge-shaped stamp on the rigid plastic surface. The length of the contact area c can be expressed in terms of the pressing depth h as

$$h = c(\sin\varphi\, \mathrm{e}^{-\cot\,(2\varphi)(\varphi-\alpha)} - \sin\alpha),$$

from which it follows that, for $\alpha < \varphi$, the pressing depth $h > 0$.

The stamp pressure force required to deform the material is given by the relation

$$P = \frac{kc}{\cos 2\varphi}\Big(1 - \frac{1-\cos 2\varphi}{1+\cos 2\varphi}\, \mathrm{e}^{-2\theta\cot\,2\varphi}\Big)\sin\alpha.$$

The projections of the velocity vector in the plastic region are written as

$$u = \frac{v_0\sin\alpha}{\cos\varphi}\, \mathrm{e}^{-\cot\,(2\varphi)(\psi-\alpha)}, \qquad v = 0,$$

and the displacement velocity components are

$$v_1 = \frac{v_0\sin\alpha}{\cos\varphi}\, \mathrm{e}^{-\cot\,(2\varphi)(\psi-\alpha)}\sin(\psi+\varphi) - v_0,$$
$$v_2 = -\frac{v_0\sin\alpha}{\cos\varphi}\, \mathrm{e}^{-\cot\,(2\varphi)(\psi-\alpha)}\cos(\psi+\varphi). \tag{2}$$

On the discontinuity line $A_0A_1A_2A_3$, the tangential and normal velocity components are equal to

$$v_t = \frac{v_0\sin\alpha\sin 2\varphi}{\cos\varphi}\, \mathrm{e}^{-\cot\,(2\varphi)(\psi-\alpha)}, \qquad v_n = -\frac{v_0\sin\alpha\cos 2\varphi}{\cos\varphi}\, \mathrm{e}^{-\cot\,(2\varphi)(\psi-\alpha)}.$$

The normal displacement velocity G of the discontinuity surface $A_0A_1A_2A_3$ is given by

$$G = \frac{1}{\sqrt{f_{,i}f_{,i}}}\frac{\partial f}{\partial t}$$

[$f(x_1, x_2, t) = 0$ is the equation of the rigid plastic boundary] and has the form

$$G = v_0\sin(\psi-\varphi), \qquad \alpha \leqslant \psi \leqslant \alpha+\theta.$$

The rate of change in the volume of the medium in passing through the plastic region is the sum of the rate of volume change in passing through the discontinuity surface $\int\limits_S (v_n + G)\,dS$ and the rate of volume change in the plastic region $\int\limits_V v_{i,i}\,dV$. Because we consider steady-state flow, the following equality holds:

$$\int\limits_S (v_n + G)\,dS + \int\limits_V v_{i,i}\,dV = 0. \tag{3}$$

Relation (3) leads to the following equation for the cone angle of the fan A_1PA_2:

$$\cos(\alpha+\theta) = \sin\alpha\tan\varphi\, \mathrm{e}^{-\cot(2\varphi)\theta}.$$

Let us estimate the strain fields in the problem considered. The curve $M_1M_2M_3M_4$ in Fig. 1 is the particle motion trajectory in the plastic region.

The motion of the medium will be described in Euler variables. Because the medium is assumed to be continuous, the functions x_i^0 are also considered continuous. On the discontinuity surface of the displacement velocities, the derivatives of these functions should satisfy the Hadamard–Thomas geometrical and kinematic compatibility conditions [8]

$$[x_{i,j}^0] = \lambda_i n_j, \qquad \Big[\frac{\partial x_i^0}{\partial t}\Big] = \lambda_i G, \tag{4}$$

where $[x_{i,j}^0] = x_{i,j}^{0+} - x_{i,j}^{0-}$, n_j are the components of the normal unit vector to the discontinuity surface, and λ_i are some functions defined on the discontinuity surface; the subscripts plus and minus denote the sides of the discontinuity surface.

Below the discontinuity line of the displacement velocities BA_2A_3, the material is assumed to be not deformed:

$$a_{ji}\Big|_{M_1} = x_{i,j}^{0-} = \delta_{ij}. \tag{5}$$

Because the Lagrangian coordinates are constant along each material particle trajectory, we have

$$\frac{dx_j^0}{dt} = \frac{\partial x_j^0}{\partial t} + v_k\,\frac{\partial x_j^0}{\partial x_k} = 0,$$

whence follows

$$\Big[\frac{\partial x_j^0}{\partial t}\Big] = -\Big[v_k\,\frac{\partial x_j^0}{\partial x_k}\Big].$$

Using the first relation in (4) and relations (5), we obtain

$$\Big[\frac{\partial x_j^0}{\partial t}\Big] = -[v_j] - \lambda_j v_{n+} \tag{6}$$

(v_{n+} is the normal particle velocity on the discontinuity line). The displacement velocity discontinuity vector can be written as

$$[v_j] = [v_t]t_j + [v_n]n_j,$$

where $[v_t]$ is the magnitude of the discontinuity of the tangential velocity component, $[v_n]$ is the magnitude of the discontinuity of the normal velocity component, and t_j are the components of the tangential unit vector to the discontinuity surface.

From a comparison of the right sides of (4) and (6), it follows that [9]:

$$\begin{gathered}[x_{i,j}^0] = -(W_1t_i + W_2n_i)n_j, \qquad a_{ji}\Big|_{M_2} = x_{i,j}^{0+} = \delta_{ij} - (W_1t_i + W_2n_i)n_j,\\ W_1 = [v_t]/(G+v_{n+}), \qquad W_2 = [v_n]/(G+v_{n+})\end{gathered} \tag{7}$$

(W_1 and W_2 are the volumetric energy densities of the shear and volumetric strains normalized by the yield limit k). From relations (7), it is possible to obtain the increments of the distorsion tensor component on the discontinuity lines BA_2A_3 and A_0A_1B.

To obtain the distorsion tensor components in the region of continuous plastic deformation (along the trajectory M_2M_3), we use the system [10]

$$\frac{da_{ij}}{dt} + a_{kj}\,\frac{\partial v_k}{\partial x_i} = 0 \qquad (k=1,2), \tag{8}$$

where $d/dt = \partial/\partial t + v_k\,\partial/\partial x_k$ is the material derivative with respect to time. For steady-state plastic deformation and a stationary velocity field, we have

$$\frac{\partial a_{ij}}{\partial t} = 0, \qquad \frac{da_{ij}}{dt} = v_k \frac{\partial a_{ij}}{\partial x_k}.$$

Transforming from the coordinates x_1 and x_2 to the curvilinear coordinates ξ and η and using the relations for the curvature radii of the characteristics R_ξ and R_η

$$\frac{1}{R_\xi} = \frac{\partial \psi}{\partial S_\xi}, \qquad \frac{1}{R_\eta} = -\frac{\partial \psi}{\partial S_\eta},$$

and the expressions for the derivatives with respect to the direction coincident with the direction of the tangent to the characteristic, we obtain

$$\begin{gathered}\frac{\partial}{\partial x_1} = \frac{2}{\sin 2\varphi}\Big(\frac{\partial}{\partial \xi}\,\frac{\sin(\psi+\varphi)}{R_\xi} + \frac{\partial}{\partial \eta}\,\frac{\sin(\psi-\varphi)}{R_\eta}\Big),\\ \frac{\partial}{\partial x_2} = -\frac{2}{\sin 2\varphi}\Big(\frac{\partial}{\partial \xi}\,\frac{\cos(\psi+\varphi)}{R_\xi} + \frac{\partial}{\partial \eta}\,\frac{\cos(\psi-\varphi)}{R_\eta}\Big),\end{gathered} \tag{9}$$

where $\psi = (\xi+\eta)/2$.

Because in the problem considered, the velocity field is expressed as (2) and the characteristics η are straight lines, by passing to the limit $R_\eta \to \infty$ in (9), we can transform system (8), by means of (9), to the system of ordinary differential equations

$$\begin{gathered}\frac{da_{11}}{d\psi}\,B + (-a_{11}\sin(\psi-\varphi) + a_{21}\cos(\psi-\varphi))\sin(\psi+\varphi) = 0,\\ \frac{da_{12}}{d\psi}\,B + (-a_{12}\sin(\psi-\varphi) + a_{22}\cos(\psi-\varphi))\sin(\psi+\varphi) = 0,\\ \frac{da_{21}}{d\psi}\,B + (a_{11}\sin(\psi-\varphi) - a_{21}\cos(\psi-\varphi))\cos(\psi+\varphi) = 0,\\ \frac{da_{22}}{d\psi}\,B + (a_{12}\sin(\psi-\varphi) - a_{22}\cos(\psi-\varphi))\cos(\psi+\varphi) = 0,\end{gathered} \tag{10}$$

$$B = \frac{\sin 2\varphi(\sin\alpha\,\mathrm{e}^{-\cot\,(2\varphi)(\psi-\alpha)} - \cos\varphi\sin(\psi+\varphi))}{\sin\alpha\,\mathrm{e}^{-\cot\,(2\varphi)(\psi-\alpha)}}.$$

System (10) describes the dependence of the distorsion tensor components on the parameter ψ distributed along the trajectory of each particle (the variable ψ plays the role of time, and $da_{ij}/d\psi$ is the convective component of the material derivative). Numerical integration of system (10) yields the components a_{ij} at the point M_3. The initial conditions for the system of differential equations (10) are the distorsion tensor components (7) calculated at the point M_2. The Almansi strain tensor is calculated from the distorsion tensor using constraints (1).

In passing from the plastic region across the discontinuity line, the particle displacement velocity A_0A_1B is deformed again. Because the strain gradient tensor for the initial, intermediate, and final configurations of the medium are linked by the relation

$$\frac{\partial x_i^0}{\partial x_j} = \frac{\partial x_i^0}{\partial x_k^{*0}}\,\frac{\partial x_k^{*0}}{\partial x_j},$$

the total strains at the point M_4 are expressed as

$$\frac{\partial x_i^{0+}}{\partial x_j}\Big|_{M_4} = (\delta_{ik} - (W_1 t_i + W_2 n_i)n_k)\,\frac{\partial x_k^{*0-}}{\partial x_j}\Big|_{M_3}. \tag{11}$$

Here

$$\frac{\partial x_k^{*0-}}{\partial x_j}\Big|_{M_3} = a_{jk}\Big|_{M_3}.$$

Using relations (7) and (11) and the system of differential equations (10), we obtain the strain accumulation in the regions M_1M_2, M_2M_3, and M_3M_4.

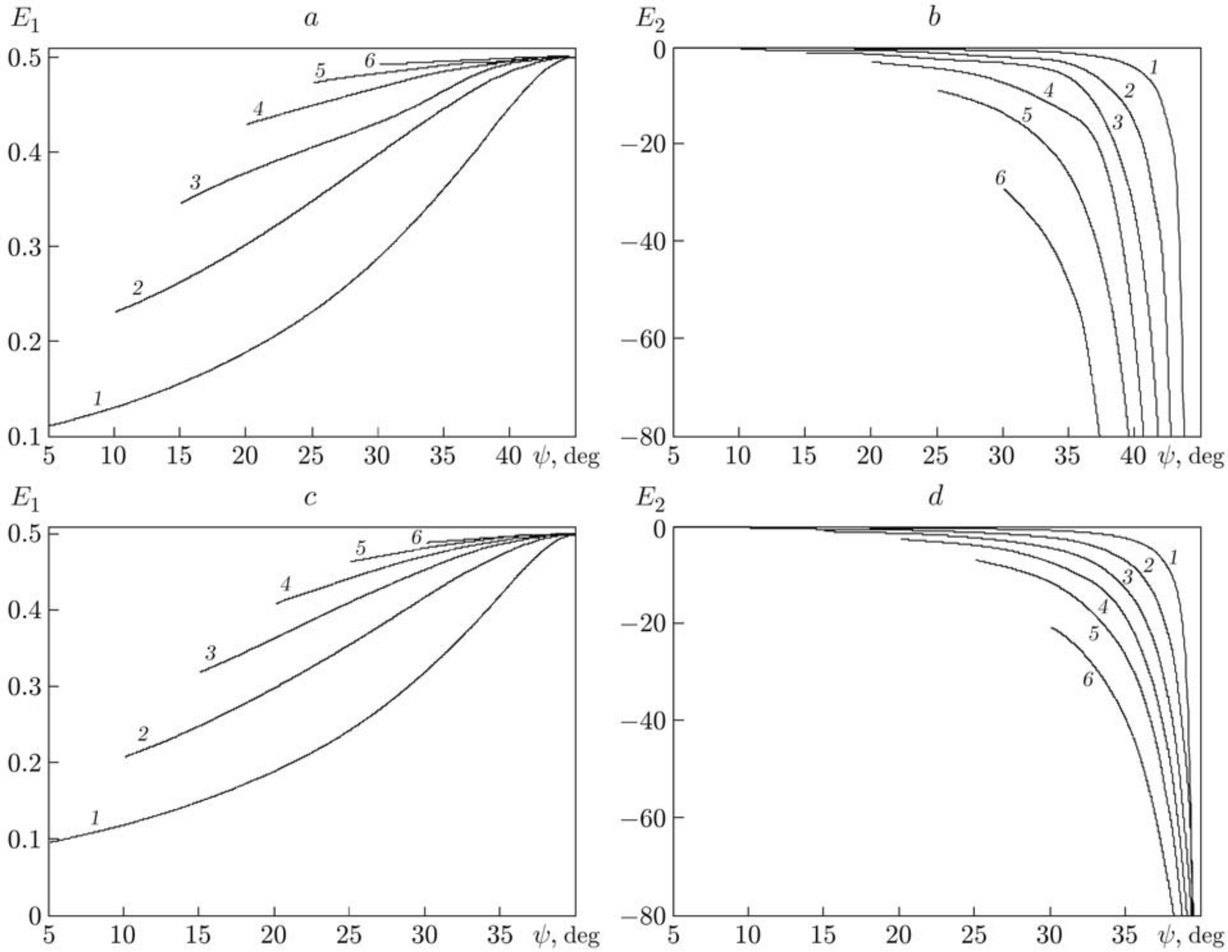

Fig. 2. Strains E_1 (a and c) and E_2 (b and d) versus angle ψ in the region CA_0A_1BD (see Fig. 1) for $\varphi = 45^\circ$ (incompressible material) (a and b) and 40° (c and d) and $\alpha = 5^\circ$ (1), 10° (2), 15° (3), 20° (4), 25° (5), and 30° (6).

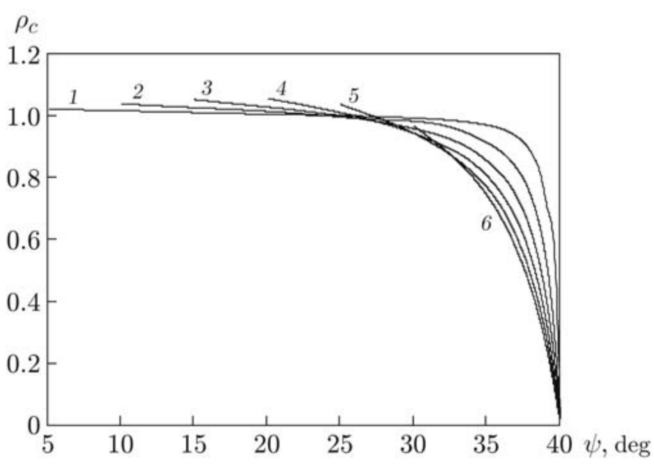

Fig. 3. Material density versus angle ψ in the region CA_0A_1BD (see Fig. 1) for $\varphi = 40^\circ$ and various values of the angle α (notation the same as in Fig. 2).

The particle strain characteristics are taken to be the principal values of the Almansi finite strain tensor (1)

$$E_{1,2} = \frac{1}{2}(E_{11} + E_{22}) \pm \frac{1}{2}\sqrt{(E_{11} - E_{22})^2 + 4E_{12}^2}.$$

The change in the material density due to deformation is given by the relation

$$\rho_c = \sqrt{(1 - 2E_1)(1 - 2E_2)}\,\rho_c^0,$$

where ρ_c^0 is the initial density.

The numerical solution of the problem was obtained for the following parameter values: $v_0 = 1$, $\rho_c^0 = 1$, and $\varphi = 40$ and 45°. Figure 2 shows curves of particle strains at the exit from the plastic region versus angle ψ ($\alpha \leqslant \psi \leqslant \varphi$) for values of the angle $\alpha = 5$–30°.

Figure 3 shows curves of the material density ρ_c at the exit from the plastic region versus angle ψ for $\varphi = 40^\circ$ and values of the angle $\alpha = 5$–30°. From Fig. 3 follows that, at $\alpha = 30^\circ$, the pressing of the medium leads to its decompaction. In the remaining cases, the material is first compacted and then decompacted in approaching the point B.

The study performed leads to the following conclusions.

The proposed model for the technological process of surface pressing provides quantitative estimates of changes in material density.

The approach can be used in experimental determinations of the constant φ in the Coulomb–Mohr yield condition.

The mechanical characteristic of material fracture related to plastic compressibility (from the moment of onset of surface peeling) can be determined by experimental implementation of the surface pressing of various materials by a wedge-shaped stamp.

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